Big Goal : study $H^k(GL_n\mathbb{Z};\mathbb{Q})$ for $k,n \geq 0$.

Aim of Talk : Describe recent results establishing strong algebraic structures on $\bigoplus_{k,n\geq 0} H^k(GL_n\mathbb{Z};\mathbb{Q})$

But first : What even is $H^k(GL_n\mathbb{Z};\mathbb{Q})$ and why do we care?

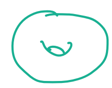
(I) Group (co)homology

Q : What is $H_k(G;M)$, $H^k(G;M)$?

G: group
$G \curvearrowright M$, M ab. group

Topologically : (Co)hom. of "classifying space of G"

$H_k(X_G;M)$, $H^k(X_G;M)$

Eg : $\mathbb{Z} \rightsquigarrow$  S^1 $\qquad$ $\mathbb{Z}\times\mathbb{Z} \rightsquigarrow$  $S^1 \times S^1$

Algebraically : – "flat resolution of M"

$$\cdots \to F_1 \to F_0 \to M \to 0$$

– Take H_* of

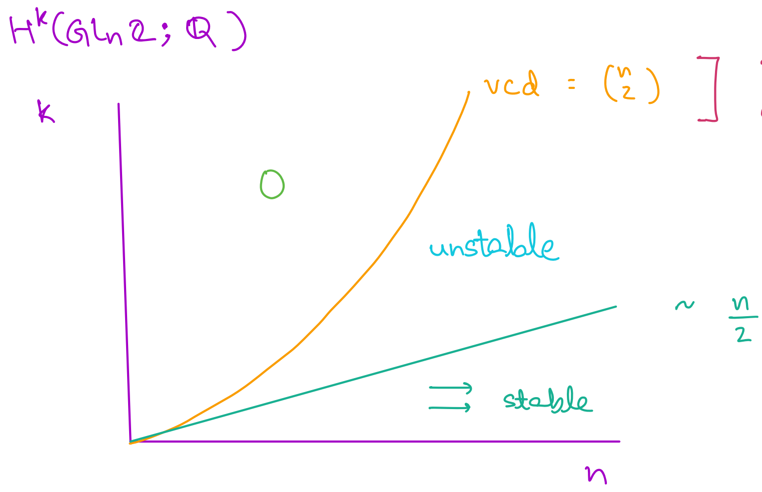
$$\cdots \to F_1 \otimes_G \mathbb{Z} \to F_0 \otimes_G \mathbb{Z} \longrightarrow 0$$

Takeaway : Can define $H_k(G;M)$, $H^k(G;M)$ algebraically and topologically

Our Focus : $H^k(GL_n\mathbb{Z};\mathbb{Q})$ $\quad k,n \geq 0$

(II) The groups $H^k(GL_n\mathbb{Z};\mathbb{Q})$

Important in – Number Theory
K- Theory
Topology



related to fact that $\exists$ classifying space model of dim $\binom{n}{2}$ ... kinda...

III Duality for $GL_n\mathbb{Z}$

- $GL_n\mathbb{Z}$ is a rational duality group ; satisfies an analogue of Poincare duality of manifolds

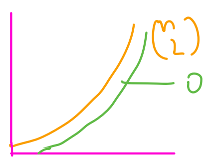
Thm : $H^k(GL_n\mathbb{Z};\mathbb{Q}) \cong H_{\binom{n}{2}-k}(GL_n\mathbb{Z};\underbrace{St_n}_{\text{Steinberg module}})$
[Borel - Serre, Bieri-Eckmann]

(will define later)

Advantages :

- High deg $H^k \rightsquigarrow$ low deg H_*.
  Can compute using partial resolutions of St_n

$$F_1 \longrightarrow F_0 \longrightarrow St_n \longrightarrow 0$$



- This approach has been used to show some of these groups are 0 -

- Algebraic structures on $St_n \rightsquigarrow$
  Algebraic structures on $\bigoplus_{k,n\geq 0} H_k(GL_n\mathbb{Z};St_n)$

— Ash - Miller - Patzt (2024) —

Thm : $\bigoplus_{k,n} H_k(GL_n\mathbb{Z};St_n)$ forms a commutative graded Hopf algebra

Remainder of Talk :
- What is St_n?
- Hopf Algebra Structure
- How this helps

IV The Steinberg Module

- Defined in terms of certain simplicial complexes $\underbrace{\mathcal{T}_n\mathbb{Q}}_{\text{Solomon-Tits buildings}}$

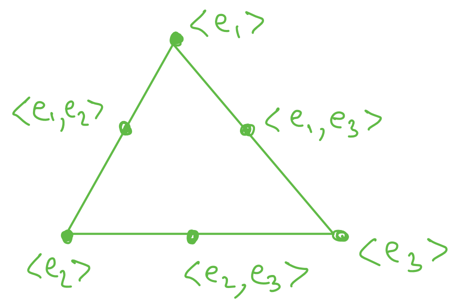
Vertices $\longleftrightarrow$ $0 \subsetneq V \subsetneq \mathbb{Q}^n$ proper, nonzero subspaces

p-simplices $\longleftrightarrow$ Flags of subspaces
$0 \subsetneq V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_p \subsetneq \mathbb{Q}^n$

Eg: n = 2 $\quad \mathcal{T}_2\mathbb{Q}$ has a vertex for every line $L \subset \mathbb{Q}^2$

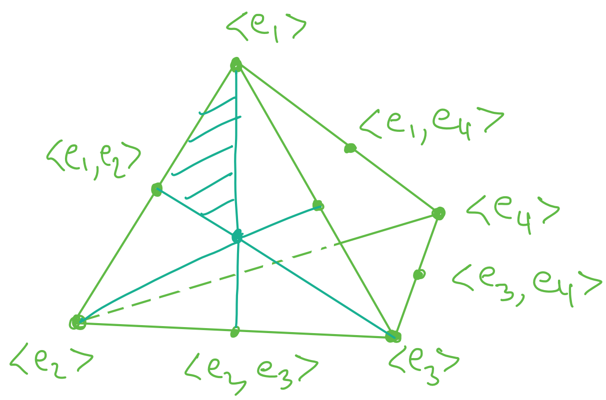
$\langle e_1 \rangle \quad \langle e_2 \rangle$] "apartment"

Eg: n = 3 $\quad \mathcal{T}_3\mathbb{Q}$ has vertices for lines & planes $L, P \subset \mathbb{Q}^3$.
Edges $\rightsquigarrow$ inclusions



"apartment"

Eg: n = 4



"apartment"

In general, apartment of $\mathcal{T}_n\mathbb{Q} \cong \partial\Delta^{n-1}$ (bdry of $(n-1)$-simplex)

In fact, $\mathcal{T}_n\mathbb{Q}$ is made out of "gluing apartments together" in a certain way.

Thm: [Solomon - Tits] $\mathcal{T}_n\mathbb{Q} \simeq \bigvee S^{n-2}$

$$\boxed{St_n := \tilde{H}_{n-2}(\mathcal{T}_n\mathbb{Q})}$$ } generated by apartment classes

Next Goal: Describe a product and coproduct on $\bigoplus_n St_n$ (analogous to multiplication and factoring in $\mathbb{N}$)

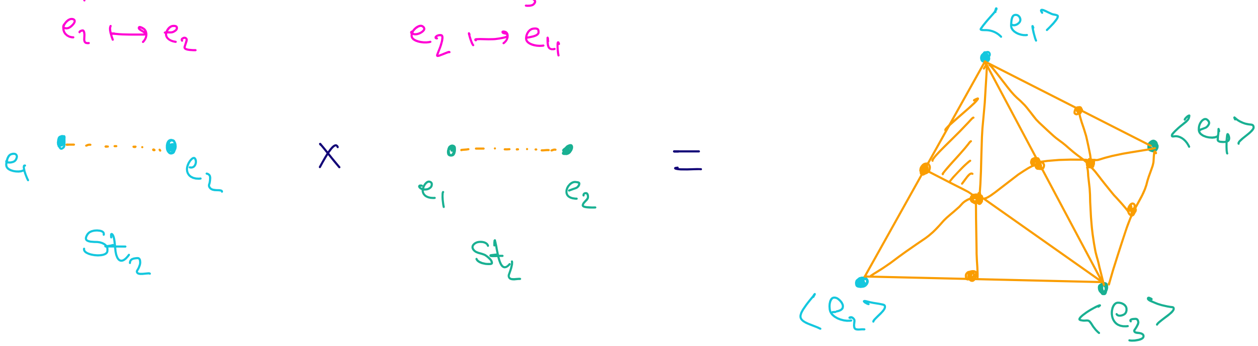
Eg: (Product)

$$\underset{St_2}{[e_1, e_2]} \times \underset{St_2}{[e_1, e_2]} = \underset{St_4}{[e_1, e_2, e_3, e_4]}$$

$$\mathbb{Q}^2 \oplus \mathbb{Q}^2 \cong \mathbb{Q}^4$$

$e_1 \mapsto e_1$ $e_1 \mapsto e_3$

$e_2 \mapsto e_2$ $e_2 \mapsto e_4$



In general, $St_n \otimes St_m \to St_{m+n}$

Formal algebraic results let us get

$$\boxed{H_k(GL_n\mathbb{Z}; St_n) \otimes H_\ell(GL_m\mathbb{Z}; St_m) \to H_{k+\ell}(GL_{m+n}\mathbb{Z}; St_{m+n})}$$

Eg: (Coproduct)

$$[e_1, e_2, e_3, e_4] \longmapsto \begin{aligned}[t] & [e_1, e_2] \otimes [e_3, e_4] \\ + \; & [e_2, e_3] \otimes [e_1, e_4] \\ + \; & \cdots \\ + \; & [e_1] \otimes [e_2, e_3, e_4] \\ + \; & \cdots \end{aligned}$$

$$St_n \longrightarrow \bigoplus_q St_q \otimes St_{n-q}$$

We get

$$\boxed{H_k(GL_n\mathbb{Z}; St_n) \longrightarrow \bigoplus_{p,q} H_p(GL_q\mathbb{Z}; St_q) \otimes H_{k-p}(GL_{n-q}\mathbb{Z}; St_{n-q})}$$

V. Hopf Algebras

The product & coproduct on $\bigoplus H_k(GL_n\mathbb{Z}; St_n)$ are compatible in the sense of a "Hopf Algebra"

Analogous to the compatibility of multiplying and factoring in $\mathbb{N}$.

Eg:

$$\begin{tikzcd} 4 \otimes 3 \arrow[r] \arrow[d] & 12 \arrow[r] & 4\otimes 3 \oplus 2\otimes 6 \oplus 3\otimes 4 \cdots \\ (2\otimes 2 \oplus 1\otimes 4 \oplus 4\otimes 1) \otimes (1\otimes 3 \oplus 3\otimes 1) \arrow[rr] & & (4\otimes 1)\otimes(1\otimes 3) \oplus (2\otimes 1)\otimes(2\otimes 3) \oplus \cdots \arrow[u] \end{tikzcd}$$

Thm: $\mathbb{N}$ is freely generated by primes (as an algebra)

A Hopf algebra in some sense generalises these notions.

Defn: H is a Hopf algebra if $\exists$ $\nabla : H \otimes H \to H$
$\Delta : H \to H \otimes H$

s.t.

$$\begin{tikzcd}
H \otimes H \arrow[r] \arrow[d] & H \arrow[r] \arrow[d, phantom, "\circlearrowleft"] & H \otimes H \\
(H \otimes H) \otimes (H \otimes H) \arrow[rr] & & (H \otimes H) \otimes (H \otimes H) \arrow[u]
\end{tikzcd}$$

Thm: [Milnor-Moore] A graded commutative Hopf algebra is freely generated by its indecomposables.

Ash-Miller-Patzt:

- Showed $\bigoplus_{k,n} H_k(GL_n\mathbb{Z}; St_n)$ is a graded commutative Hopf algebra
- Found some indecomposables, and used them to get new H_*-classes by multiplying them.